Journal of Computational Physics 228 (2009) 109–122

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

# A splitting extrapolation for solving nonlinear elliptic equations with d-quadratic finite elements ☆

Yong Cao [a], Xiaoming He [b,*], Tao Lü [c]

[a] Department of Mechanical Engineering and Automation, Harbin Institute of Technology Shenzhen Graduate School, Shenzhen, Guangdong 518055, PR China
[b] Department of Mathematics, Virginia Tech, MCB 460, Blacksburg, VA 24061, USA
[c] Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China

ARTICLE INFO

*Article history:*
Received 15 April 2008
Received in revised form 26 July 2008
Accepted 2 September 2008
Available online 12 September 2008

*AMS subject classification:*
65B05
65N12
65N30
35J60

*Keywords:*
Extrapolation
Asymptotic expansion
Parallel algorithm
Finite elements
Domain decomposition
A posteriori error estimate

ABSTRACT

Nonlinear elliptic partial differential equations are important to many large scale engineering and science problems. For this kind of equations, this article discusses a splitting extrapolation which possesses a high order of accuracy, a high degree of parallelism, less computational complexity and more flexibility than Richardson extrapolation. According to the problems, some domain decompositions are constructed and some independent mesh parameters are designed. Multi-parameter asymptotic expansions are proved for the errors of approximations. Based on the expansions, splitting extrapolation formulas are developed to compute approximations with high order of accuracy on a globally fine grid. Because these formulas only require us to solve a set of smaller discrete subproblems on different coarser grids in parallel instead of on the globally fine grid, a large scale multidimensional problem is turned into a set of smaller discrete subproblems. Additionally, this method is efficient for solving interface problems.



## 1. Introduction

In this paper, we consider a finite element splitting extrapolation for solving the following modeling nonlinear elliptic equations:

$$\begin{cases} Lu = -\sum_{i,j=1}^{d} D_i(a_{ij}(x,u)D_j u) = f(x,u) & \text{on } \Omega, \\ u = g(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \Re^d (d = 2, 3), a_{ij}(x,u) \in L_\infty(\Omega), x = (x_1, \ldots, x_d), D_i = \frac{\partial}{\partial x_i}$. Without loss of generalization, we only consider the equations with Dirichlet boundary condition here. The splitting extrapolation can be used to handle the equations with other boundary conditions similarly.

☆ This work is partially supported by NSF Grant DMS-0713763 and the National Science Foundation of China (10671136, 10875034).
* Corresponding author. Tel.: +1 540 239 7233.
*E-mail addresses:* yongc@hitsz.edu.cn (Y. Cao), xiaoming@math.vt.edu (X. He), lutao@public.cd.sc.cn (T. Lü).

0021-9991/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2008.09.003

It is well known that efficiently and accurately solving this problem is critical in many applications of engineering and sciences, such as heat and mass transfer phenomena and electrostatic field problems. For example, in inhomogeneous and/or anisotropic media, the thermal conductivity (diffusion coefficient) can depend on the coordinates, the temperature and the heat transfer direction. Recently, many efforts have been attempted to describe the thermal conductivity more accurately, see [5,7] and reference there in. However, no matter how the thermal conductivity is described, for the phenomena in steady state, an inhomogeneous and/or anisotropic medium always leads to our modeling nonlinear elliptic partial differential equation.

The modeling problem can be solved by conventional numerical methods, including both finite difference (FD) methods and finite element (FE) methods. However, for large scale problem, how to solve this modeling problem more accurately and efficiently still remains challenging. Richardson extrapolation is an efficient acceleration method to improve the accuracy and the rate of convergence. Therefore, a lot of work was contributed to this method, see [1,2,6,11,14,17,23,25,29] and reference therein.

Nevertheless, unbalanced loads is a shortcoming of Richardson extrapolation method on parallel algorithm. In addition, Richardson extrapolation has high complexity order and some strict smoothness requirements for the analytic solutions. Splitting extrapolation is developed by Lin and Lü [15] in 1983 to get rid of these limitations. First, we design some independent mesh parameters, say, $h_1, \dots, h_k$, and let $u(h_1, \dots, h_k)$ to denote the corresponding approximation. Once we prove a multi-parameter asymptotic expansion of the error for the independent parameters, we can follow the way in Section 4 to construct a special linear combination of $u(h_1, \dots, h_k)$, $u(\frac{h_1}{2}, \dots, h_k), \dots, u(h_1, \dots, \frac{h_k}{2})$, so that a new approximation with higher order of accuracy is obtained.

This method is naturally parallel with high degree of parallelism, improves the accuracy with less computational complexity than Richardson extrapolation and only requires piecewise smoothness for the analytic solutions. The design of the independent parameters also gives us flexibility in choosing different kinds of meshes. In addition, it can save a lot of memory if we want to use sequential computation. These advantages of splitting extrapolation become more clear and powerful when the size of the problem is large and more independent meshes sizes are designed with domain decomposition. For more background, we refer the reader to [10,13,15,16,21,22,24,27,28].

Finite element splitting extrapolation based on domain decomposition is an important development of splitting extrapolation. First an initial domain decomposition is constructed according to the dimension and interface of the problem, the shape and size of the domain, and the computers used. Then the independent parameters are designed for all subdomains. The algorithm combines advantages of domain decomposition and splitting extrapolation and can be applied to interface problems with discontinuous coefficients. After Lü [18] proposed the idea in 1987, the finite element splitting extrapolation based on domain decomposition and linear finite elements has been presented in [12,13,19,30]. Lü et al. [9,20] have presented a finite element splitting extrapolation based on domain decomposition and d-quadratic iso-parametric finite elements to solve linear elliptic and parabolic equations with curved boundaries. Obviously, the splitting extrapolation of quadratic finite elements possesses higher order of accuracy than that of linear finite elements.

Since this kind of finite element splitting extrapolation is efficient for linear problems, it is hopeful to apply it to nonlinear problems. However, the analysis for nonlinear cases is much different from and more difficult than that of the linear cases. Hence, it is not trivial to extend this method for nonlinear problems. In this paper, we will investigate the finite element splitting extrapolation based on domain decomposition and d-quadratic iso-parametric finite elements for solving second order nonlinear elliptic equations with curved boundaries.

This paper is organized as follows: in Section 2, we will introduce some preliminaries and notations; in Section 3, we will prove the multi-variable asymptotic expansion of d-quadratic iso-parametric finite element errors; in Section 4, we will develop the corresponding splitting extrapolation formulas; in Section 5, we will introduce a parallel/sequential algorithm; in Section 6, we will present some a posteriori error estimates; in Section 7, we present two numerical examples.

## 2. Some preliminaries and notations

First, with the same method as in [4,9,20], we construct the partition and d-quadratic iso-parametric mapping as follows. Even though the following set-up is well known, we still repeat it here in order to introduce the notations that will be used. We construct a non-overlapping initial domain decomposition $\overline{\Omega} = \bigcup_{i=1}^{m} \overline{\Omega}_i$, where $\Omega_i (i = 1, \dots, m)$ are allowed to have some curved boundaries, but satisfy the compatibility condition, i.e., $\overline{\Omega}_i \bigcap \overline{\Omega}_j (i \neq j)$ is either empty or the set of common vertices, common edges and common surfaces. $\partial\Omega_i \setminus \partial\Omega$ is called a pseudo-boundary, which is often the interface of a discontinuous coefficient function. For the initial partition, we assume the following:

(1) There are unit cubes $\widehat{\Omega}_i (i = 1, \dots, m) \subset \Re^d$ and one-to-one d-quadratic iso-parametric mappings $\Psi_i : \Omega_i \to \widehat{\Omega}_i$ such that $\{\Psi_i^{-1}\}$ are sufficiently smooth.
(2) $\overline{\widehat{\Omega}} = \bigcup_{i=1}^{m} \overline{\widehat{\Omega}}_i$ is an initial uniform partition satisfying the compatibility condition.
(3) If $\Gamma_{ij} = \partial\Omega_i \bigcap \partial\Omega_j$, then $\Psi_i(x) = \Psi_j(x), \forall x \in \Gamma_{ij}$.
Let $\widehat{\Im}_i^h$ $(i = 1, \dots, m)$ be a uniform cuboid partition with grid parameter $\hat{h}_{ij}$ $(j = 1, \dots, d)$ on $\widehat{\Omega}_i$ such that $\widehat{\Im}^h = \bigcup_{i=1}^{m} \widehat{\Im}_i^h$ is a piecewise uniform cuboid partition on $\widehat{\Omega}$. Note that because of compatibility conditions, there are only $l$ $(l < md)$ independent grid parameters which are denoted by $\hat{h}_1, \dots, \hat{h}_l$. From the construction of mappings $\{\Psi_i\}$, we can derive a compatible partition $\Im^h = \bigcup_{i=1}^{m} \Im_i^h$ on $\Omega$. We also have that

(4) If $\hat{z}$ is an inner grid point in $\widehat{\Omega}_i$, then $z = \Psi_i^{-1}(\hat{z})$ is also an inner grid point in $\Omega_i$.
(5) If $\hat{z} \in \widehat{\Gamma}_{ij} = \partial\widehat{\Omega}_i \bigcap \partial\widehat{\Omega}_j$ is a grid point on $\widehat{\Gamma}_{ij}$, then $z = \Psi_i^{-1}(\hat{z})$ is also a grid point on $\Gamma_{ij}$.
(6) If $\hat{z} \in \partial\widehat{\Omega} \bigcap \widehat{\Omega}_i$ is a grid point on $\partial\widehat{\Omega}$, then $z = \Psi_i^{-1}(\hat{z})$ is also a grid point on $\partial\Omega$.
(7) If an element $\hat{e} \in \widehat{\Im}_i^h$, then $e = \Psi_i^{-1}(\hat{e}) \in \Im_i^h$.

By the d-quadratic iso-parametric mapping, (1.1) is converted to the following problem:

$$\begin{cases} -\sum\limits_{i,j=1}^{d} \widehat{D}_i(\hat{a}_{ij}(\hat{x},\hat{u})D_j\hat{u}) = \hat{f}(\hat{x},\hat{u}) & \text{on } \widehat{\Omega}, \\ \hat{u} = \hat{g}(\hat{x}) & \text{on } \partial\widehat{\Omega}. \end{cases} \tag{2.2}$$

where $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_d)$, $\widehat{D}_i = \frac{\partial}{\partial \hat{x}_i}$ and $\hat{u}|_{\widehat{\Omega}_i} = u \circ \Psi_i|_{\Omega_i}$.

Second, in order to ensure the existence, uniqueness and smoothness of the solution to the problem (2.2), we assume that

$$\exists \mu > 0, \sum_{i,j=1}^{d} \hat{a}_{ij}(\hat{x},\hat{u})\xi_i\xi_j \geqslant \mu \sum_{i=1}^{d} \xi_i^2, \quad \forall \hat{x} \in \widehat{\Omega}, \quad \forall \hat{u} \in \mathbf{R}. \tag{2.3}$$

Third, let $H_0^1(\widehat{\Omega}) := \{\hat{u} \in H^1(\widehat{\Omega}) : \hat{u} = 0 \text{ on } \partial\widehat{\Omega}\}$, $(\hat{u},\hat{v}) = \int_{\widehat{\Omega}} \hat{u}\hat{v}\,d\hat{x}$, then the weak form can be obtained as follows: find $\hat{u} \in H_0^1(\widehat{\Omega})$ satisfying

$$A(\hat{u},\hat{v}) = (\hat{f}(\hat{u}),\hat{v}), \forall \hat{v} \in H_0^1(\widehat{\Omega}), \tag{2.4}$$

$$A(\hat{u},\hat{v}) = \sum_{i,j=1}^{d} (\hat{a}_{ij}(\hat{x},\hat{u})\widehat{D}_i\hat{u}, \widehat{D}_i\hat{v}). \tag{2.5}$$

Fourth, let $\widehat{S}_0^h \subset H_0^1(\widehat{\Omega}) \cap C(\widehat{\Omega})$ denote the d-quadratic finite element space under the partition $\widehat{\Im}^h$, then the discrete scheme can be obtained as follows: find $\hat{u}_h \in \widehat{S}_0^h$ satisfying

$$A(\hat{u}_h,\hat{v}_h) = (\hat{f}(\hat{u}_h),\hat{v}_h), \quad \forall \hat{v}_h \in \widehat{S}_0^h. \tag{2.6}$$

It is well known that the standard Galerkin method with d-quadratic finite elements can be used to get an approximation and Richardson extrapolation can be applied to the approximation. However, our final goal here is to develop the splitting extrapolation to get an approximation of high accuracy by computing a set of smaller discrete subproblems in parallel. Therefore, as we mentioned in Section 1, we need to prove the multi-parameter asymptotic expansion of the errors of the d-quadratic iso-parametric finite element approximations and construct the splitting extrapolation formulas, which will be done in the next two sections. Here are some more conventions used in this article.

$\hat{h} := (\hat{h}_1, \ldots, \hat{h}_l)$, $\hat{h}_0 := \max_{1\leqslant i\leqslant l}\hat{h}_i$.
Coarse grid: the grid obtained from $\hat{h}^{(0)} = (\hat{h}_1, \ldots, \hat{h}_l)$.
Locally fine grid: the grid obtained from $\hat{h}^{(i)} = (\hat{h}_1, \ldots, \frac{\hat{h}_i}{2}, \ldots, \hat{h}_l)$, $i = 1, \ldots, l$.
Globally fine grid: the grid obtained from $\frac{\hat{h}^{(0)}}{2}$.
$\widehat{\Omega}_0^h$: the set of grid points obtained from $\hat{h}^{(0)} = (\hat{h}_1, \ldots, \hat{h}_l)$.
$\widehat{\Omega}_i^h$: the set of grid points obtained from $\hat{h}^{(i)} = (\hat{h}_1, \ldots, \frac{\hat{h}_i}{2}, \ldots, \hat{h}_l)$, $i = 1, \ldots, l$.

Let $\hat{u}^I$ be the finite element interpolation function of $\hat{u}$ in $\widehat{S}_0^h$. Let $\|\cdot\|_{k,p,\widehat{\Omega}}$ denote the norm of the space $W_p^k(\widehat{\Omega})$, $\|\cdot\|_{k,\widehat{\Omega}}$ denote the norm of the space $H^k(\widehat{\Omega})$ and $\|\cdot\|_{k,\infty,\widehat{\Omega}}$ denote the norm of the space $W_\infty^k(\widehat{\Omega})$. We define $\prod_{s=1}^m W_p^k(\widehat{\Omega}_s)$ to be a product space with the norm $\|\cdot\|'_{k,p,\widehat{\Omega}} := (\sum_{s=1}^m \|\cdot\|^p_{k,p,\widehat{\Omega}_s})^{\frac{1}{p}}$, $\prod_{s=1}^m H^k(\widehat{\Omega}_s)$ to be a product space with the norm $\|\cdot\|'_{k,\widehat{\Omega}} := (\sum_{s=1}^m \|\cdot\|^2_{k,\widehat{\Omega}_s})^{\frac{1}{2}}$, and $\prod_{s=1}^m W_\infty^k(\widehat{\Omega}_s)$ to be a product space with the norm $\|\cdot\|'_{k,\infty,\widehat{\Omega}} := \sup_{1\leqslant s\leqslant m}\|\cdot\|_{k,\infty,\widehat{\Omega}_s}$.

## 3. Multi-parameter asymptotic expansion of the d-quadratic iso-parametric finite element error

We presented the discrete d-quadratic iso-parametric finite element approximation for nonlinear elliptic equations in Section 2. In this section we will prove the multi-parameter asymptotic expansion of its error. Unless otherwise specified, we use $C$ to represent a generic constant $C$ whose values might be different from line to line. First, we recall the following two lemmas from [20].

**Lemma 3.1.** *Consider a linear elliptic weak form*

$$\sum_{i,j=1}^{d} (\hat{e}_{ij}(\hat{x})\widehat{D}_i\hat{w}, \widehat{D}_i\hat{v}) + (\hat{p}\hat{w},\hat{v}) = (\hat{f},\hat{v}), \quad \forall \hat{v} \in H_0^1(\widehat{\Omega})$$

*and the corresponding d-quadratic iso-parametric finite element discrete scheme*

$$\sum_{i,j=1}^{d}(\hat{e}_{ij}(\hat{x})\widehat{D}_i\hat{w}_h,\widehat{D}_i\hat{v}_h)+(\hat{p}\hat{w}_h,\hat{v}_h)=(\hat{f},\hat{v}_h),\quad \forall\hat{v}_h\in\widehat{S}_0^h.$$

*Assume that* $\hat{e}_{ij},\hat{p}\in(\prod_{s=1}^{m}W_\infty^4(\widehat{\Omega}_s))\bigcap L_\infty(\widehat{\Omega})$ *and* $\hat{w}\in(\prod_{s=1}^{m}H^7(\widehat{\Omega}_s))\bigcap H_0^1(\widehat{\Omega})$*, then there exist functions* $\hat{\phi}_i\in(\prod_{s=1}^{m}H^r(\widehat{\Omega}_s))\bigcap L^\infty(\widehat{\Omega})(i=1,\ldots,l)$ *independent of* $\hat{h}$ *such that*

$$\hat{w}^h-\hat{w}^I=\sum_{i=1}^{l}\hat{h}_i^4\hat{\phi}_i^I+\varepsilon,$$

$$\|\varepsilon\|'_{0,\infty,\widehat{\Omega}}=O\left(\hat{h}_0^{4+\alpha}|\ln\hat{h}_0|^{\frac{d-1}{d}}\right),\quad \alpha=\min(r,2)-\frac{d}{2}>0.$$

**Lemma 3.2.** *Let e be an element in* $\hat{\mathfrak{I}}^h$. *If* $\hat{u}\in W_p^6(e),\hat{q}\in W_\infty^4(e),\hat{\phi}\in Q_2(e)$*, then*

$$\int_e\hat{q}(\hat{u}-\hat{u}^I)\hat{\phi}\,\mathrm{d}\hat{x}=\sum_{i=1}^{d}\hat{h}_{ie}^4\int_e\left[\frac{1}{480}\hat{q}\hat{\phi}\widehat{D}_i^4\hat{u}-\frac{1}{45}\widehat{D}_i(\hat{q}\hat{\phi})\widehat{D}_i^3\hat{u}\right]\mathrm{d}\hat{x}+R$$

$$|R|\leqslant C(q)\hat{h}_{00}^6\|\hat{u}\|_{6,p,e}\|\hat{\phi}\|_{2,q,e},\hat{h}_{00}=\max_{1\leqslant i\leqslant d}\hat{h}_{ie},\frac{1}{p}+\frac{1}{q}=1.$$

*Here* $Q_2(e)$ *is the set of all d-quadratic polynomials on e.*

Second, we recall the definitions of the regularized Dirac function $\delta^z$, the regularized Green's function $G^z$ and the discrete Green's function $G_h^z$ from [2,3,8,26].

(1) $\forall$ point $z\in K^z$ where $K^z\in\hat{\mathfrak{I}}^h$, the regularized Dirac function $\delta^z\in C_0^\infty(K^z)$ is an approximation to the Dirac functional in $z$ which satisfies

$$\int_\Omega\delta^z(x)\hat{v}_h(x)\mathrm{d}x=\hat{v}_h(z),\quad\forall\hat{v}_h\in\widehat{S}_0^h,\tag{3.7}$$

$$\|\nabla_k\delta^z\|_{0,\infty,\widehat{\Omega}}\leqslant C\hat{h}_0^{-d-k},k=1,2,\ldots\tag{3.8}$$

(2) The regularized Green's function $G^z\in H_0^1(\widehat{\Omega})$ is defined by

$$E(G^z,\hat{v})=\hat{v}(z)=(\delta^z,\hat{v}),\quad\forall v\in H_0^1(\widehat{\Omega}),$$

where $E(\cdot,\cdot)$ is a bounded and coercive bilinear form.

(3) The discrete Green's function $G_h^z\in\widehat{S}_0^h$ satisfying

$$E(G_h^z,\hat{v}_h)=\hat{v}_h(z)=(\delta^z,\hat{v}_h),\quad\forall\hat{v}_h\in\widehat{S}_0^h.$$

Third, based on [2,8,31], we have the following estimate for the discrete Green's function, which will be used for a key step in the proof of the multi-variable asymptotic expansion.

**Lemma 3.3.**

$$\|G_h^z\|'_{2,1,\widehat{\Omega}}\leqslant C|\ln\hat{h}_0|\text{ if }\hat{h}_0\leqslant1-\eta\text{ for some }\eta\text{ with }0<\eta<1.$$

**Proof.** Using the definitions above, we have

$$\begin{cases}LG^z=\delta^z & \text{on }\widehat{\Omega},\\ G^z=0 & \text{on }\partial\widehat{\Omega}.\end{cases}\tag{3.9}$$

From the elliptic regularity estimate, which is also an a priori estimate (see [3,31] and references therein), we have

$$\|G^z\|'_{2,p,\widehat{\Omega}}\leqslant\frac{C}{p-1}\|LG^z\|'_{0,p,\widehat{\Omega}},\quad\forall p=1+\epsilon,\epsilon>0\text{ is small}.$$

Then (3.8), (3.9) and $\delta^z\in C_0^\infty(K^z)$ lead to

$$\|G^z\|'_{2,p,\widehat{\Omega}}\leqslant\frac{C}{p-1}\|\delta^z\|'_{0,p,\widehat{\Omega}}\leqslant\frac{C}{p-1}\|\delta^z\|'_{0,\infty,\widehat{\Omega}}\left(\int_{K_z}1\mathrm{d}x\right)^{\frac{1}{p}}\leqslant C\frac{q}{p}\hat{h}_0^{-d}\hat{h}_0^{\frac{d}{p}}=C\frac{q}{p}\hat{h}_0^{-d/q},$$

where $1/p + 1/q = 1$. Let $q = |\ln \hat{h}_0|$, then $p = 1 + \frac{1}{|\ln \hat{h}_0|-1}$. When $\hat{h}_0 \leqslant 1 - \eta$ for a $0 < \eta < 1$, there exists a $C$ independent of $\hat{h}_0$ such that $0 < \frac{1}{|\ln \hat{h}_0|-1} \leqslant C$. Then we have

$$\|G^z\|'_{2,1,\widehat{\Omega}} \leqslant C\|G^z\|'_{2,p,\widehat{\Omega}} \leqslant C|\ln \hat{h}_0|. \tag{3.10}$$

Now we recall the following three estimates from [2,8],

$$\|\nabla G^z\|'_{0,2,\widehat{\Omega}} + \|\nabla^2 G^z\|'_{0,1,\widehat{\Omega}} \leqslant C|\ln \hat{h}_0|,$$

$$\|\nabla(G^z - G^z_h)\|'_{0,1,\widehat{\Omega}} \leqslant C\hat{h}_0|\ln \hat{h}_0|,$$

$$\|(G^z - G^z_h)\|'_{s,2,\widehat{\Omega}} \leqslant C\hat{h}_0^{1-s}, \quad s = 0, 1.$$

Then, we have

$$\|G^z - G^z_h\|'_{1,1,\widehat{\Omega}} \leqslant \|G^z - G^z_h\|'_{0,1,\widehat{\Omega}} + |G^z - G^z_h|'_{1,1,\widehat{\Omega}} \leqslant C(\|G^z - G^z_h\|'_{0,2,\widehat{\Omega}} + \|\nabla(G^z - G^z_h)\|'_{0,1,\widehat{\Omega}}) \leqslant C\hat{h}_0|\ln \hat{h}_0|.$$

Let $I_h G^z$ be the standard finite element interpolation of $G^z$ in $\widehat{S}^h_0$, then by inverse estimate, finite element interpolation error estimate and (3.10), we get

$$\begin{aligned}\|G^z - G^z_h\|'_{2,1,\widehat{\Omega}} &\leqslant \|G^z - I_h G^z\|'_{2,1,\widehat{\Omega}} + \|I_h G^z - G^z_h\|'_{2,1,\widehat{\Omega}} \leqslant C\|G^z\|'_{2,1,\widehat{\Omega}} + C\hat{h}_0^{-1}\|I_h G^z - G^z_h\|'_{1,1,\widehat{\Omega}} \\ &\leqslant C\|G^z\|'_{2,1,\widehat{\Omega}} + C\hat{h}_0^{-1}(\|I_h G^z - G^z\|'_{1,1,\widehat{\Omega}} + \|G^z - G^z_h\|'_{1,1,\widehat{\Omega}}) \leqslant C|\ln \hat{h}_0|.\end{aligned}$$

Hence,

$$\|G^z_h\|'_{2,1,,\widehat{\Omega}} \leqslant \|G^z\|'_{2,1,\widehat{\Omega}} + \|G^z_h - G^z\|'_{2,1,\widehat{\Omega}} \leqslant C|\ln \hat{h}_0|. \qquad \square$$

Finally, with the same idea as in [19], we can prove the following multi-variable asymptotic expansion by using those three lemmas above. Note that we need to use the Lemma 3.3 to deal with the $W^2_1(\widehat{\Omega})$ norm estimate of the discrete Green's function, which is different from [19].

**Theorem 3.1.** *Along with the assumption of (2.3), if $f(\hat{u}, \hat{x})$ and $\hat{a}_{ij}(\hat{u}, \hat{x})$ is differentiable with respect to $\hat{u}$ and*

$$\hat{u} \in \prod_{s=1}^{m} W^6_\infty(\widehat{\Omega}_s) \bigcap H^1_0(\widehat{\Omega}), \tag{3.11}$$

$$-\frac{1}{2}\sum_{i,j=1}^{d} \widehat{D}_j(\hat{a}_{ij}(\hat{u}, \hat{x})\widehat{D}_i\hat{u}) - \hat{f}'(\hat{u}) \geqslant 0, \ a.e. \text{ on } \widehat{\Omega}, \tag{3.12}$$

*then the error of the solution to (2.6) satisfies the following multi-parameter asymptotic expansion:*

$$\hat{u}_h - \hat{u}^I = \sum_{k=1}^{l} \hat{\psi}^I_k \hat{h}^4_k + \hat{\varepsilon} \tag{3.13}$$

*where $\hat{\psi}_k \in H^1_0(\widehat{\Omega})$ are some functions independent on $\hat{h}$. Hence,*

$$\hat{u}_h(\widehat{X}) - \hat{u}^I(\widehat{X}) = \sum_{k=1}^{l} \hat{\psi}^I_k(\widehat{X})\hat{h}^4_k + \hat{\varepsilon}(\widehat{X}), \quad \forall \widehat{X} \in \widehat{\Omega}^h_0.$$

*If $\hat{\psi}_k \in W^r_p(\widehat{\Omega}), r - \frac{d}{p} > 0$, then*

$$\|\hat{\varepsilon}\|'_{0,\infty,\widehat{\Omega}} = O\left(\hat{h}_0^{4+\beta}|\ln \hat{h}_0|^{\frac{2d-1}{d}}\right), \beta = \min\left(1, \alpha, r - \frac{d}{p}\right).$$

**Proof.** First, we define a new bilinear form

$$\widetilde{A}(\hat{w}, \hat{v}) = \sum_{i,j=1}^{d} (\hat{a}_{ij}(\hat{u})\widehat{D}_i\hat{w}, \widehat{D}_i\hat{v}), \quad \forall \hat{w}, \hat{v} \in H^1_0(\widehat{\Omega}). \tag{3.14}$$

Using (2.5) and (3.14), we get

$$\widetilde{A}(\hat{u},\hat{v}) = A(\hat{u},\hat{v}), \quad \forall \hat{u},\hat{v} \in H_0^1(\widehat{\Omega}). \tag{3.15}$$

Second, let $\widehat{R}_h : H_0^1(\widehat{\Omega}) \to \widehat{S}_0^h$ denote the Ritz projection operator with respect to $\widetilde{A}(\cdot,\cdot)$, i.e.,

$$\widetilde{A}(\widehat{R}_h\hat{w},\hat{v}) = \widetilde{A}(\hat{w},\hat{v}), \quad \forall \hat{w},\hat{v} \in H_0^1(\widehat{\Omega}). \tag{3.16}$$

Let $\tilde{\hat{u}}_h = \widehat{R}_h\hat{u}$, then $\tilde{\hat{u}}_h \in \widehat{S}_0^h$. By the definition of $\tilde{\hat{u}}_h$, (2.4), (3.15) and (3.16), we get

$$\widetilde{A}(\tilde{\hat{u}}_h,\hat{\psi}_h) = (\hat{f}(\hat{u}),\hat{\psi}_h), \quad \forall \hat{\psi}_h \in \widehat{S}_0^h, \tag{3.17}$$

$$\sum_{i,j=1}^{d} (\hat{a}_{ij}(\hat{u})\widehat{D}_i\tilde{\hat{u}}_h, \widehat{D}_i\hat{\psi}_h) = (\hat{f}(\hat{u}),\hat{\psi}_h), \quad \forall \hat{\psi}_h \in \widehat{S}_0^h. \tag{3.18}$$

Let $\hat{\theta}_h = \hat{u}_h - \tilde{\hat{u}}_h$ and $\hat{\rho}_1 = \tilde{\hat{u}}_h - \hat{u}^I$, then

$$\hat{u}_h - \hat{u}^I = \hat{\theta}_h + \hat{\rho}_1. \tag{3.19}$$

Since (3.18) is a linear discrete scheme for $\tilde{\hat{u}}_h$, then we can apply Lemma 3.1 to (3.18), so we have

$$\hat{\rho}_1 = \tilde{\hat{u}}_h - \hat{u}^I = \sum_{k=1}^{l} \hat{h}_k^4 \hat{\phi}_k^I + \hat{\eta}_1, \tag{3.20}$$

$$\|\hat{\eta}_1\|'_{0,\infty,\widehat{\Omega}} = O\left(\hat{h}_0^{4+\alpha}|\ln \hat{h}_0|^{\frac{d-1}{d}}\right), \quad \alpha = \min(r,2) - \frac{d}{2} > 0. \tag{3.21}$$

Now we discuss the expansion of $\hat{\theta}$ in detail as follows. Using (2.5), (2.6) and (3.14), we get

$$\begin{aligned}\widetilde{A}(\hat{u}_h,\hat{\psi}_h) &= \sum_{i,j=1}^{d} (\hat{a}_{ij}(\hat{u})\widehat{D}_i\hat{u}_h, \widehat{D}_i\hat{\psi}_h) \\ &= \sum_{i,j=1}^{d} ((\hat{a}_{ij}(\hat{u}) - \hat{a}_{ij}(\hat{u}_h))\widehat{D}_i\hat{u}_h, \widehat{D}_j\hat{\psi}_h) + (\hat{f}(\hat{u}_h),\hat{\psi}_h) \quad \forall \hat{\psi}_h \in \widehat{S}_0^h.\end{aligned} \tag{3.22}$$

Then (3.17), (3.22) and the Taylor expansion lead to

$$\begin{aligned}\widetilde{A}(\hat{\theta}_h,\hat{\psi}_h) &= (\hat{f}(\hat{u}_h) - \hat{f}(\hat{u}),\hat{\psi}_h) + \sum_{i,j=1}^{d} ((\hat{a}_{ij}(\hat{u}) - \hat{a}_{ij}(\hat{u}_h))\widehat{D}_i\hat{u}_h, \widehat{D}_j\hat{\psi}_h) \\ &= (\hat{f}'(\hat{u})(\hat{u}_h - \hat{u}),\hat{\psi}_h) + (\hat{\varepsilon}_0,\hat{\psi}_h) - \sum_{i,j=1}^{d} (\hat{a}'_{ij}(\hat{u})(\hat{u}_h - \hat{u})\widehat{D}_i\hat{u}_h, \widehat{D}_j\hat{\psi}_h) + \sum_{j=1}^{d} (\hat{\varepsilon}_j, \widehat{D}_j\hat{\psi}_h) \\ &= (\hat{f}'(\hat{u})(\hat{u}_h - \hat{u}),\hat{\psi}_h) - \sum_{i,j=1}^{d} (\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}(\hat{u}_h - \hat{u}), \widehat{D}_j\hat{\psi}_h) + \sum_{j=1}^{d} (\hat{\delta}_j, \widehat{D}_j\hat{\psi}_h) + (\hat{\varepsilon}_0,\hat{\psi}_h) + \sum_{j=1}^{d} (\hat{\varepsilon}_j, \widehat{D}_j\hat{\psi}_h), \quad \forall \hat{\psi}_h \in \widehat{S}_0^h\end{aligned} \tag{3.23}$$

where

$$\hat{\varepsilon}_0 = \hat{f}''(\hat{\xi}_1)(\hat{u}_h - \hat{u})^2, \hat{\xi}_1 \text{ is between } \hat{u} \text{ and } \hat{u}_h,$$

$$\hat{\varepsilon}_j = \sum_{i=1}^{d} -\hat{a}''_{ij}(\hat{\xi}_{ij})(\hat{u}_h - \hat{u})^2\widehat{D}_i\hat{u}_h, \hat{\xi}_{ij} \text{ is between } \hat{u} \text{ and } \hat{u}_h, \quad j = 1,\ldots,d,$$

$$\hat{\delta}_j = -\sum_{i=1}^{d} \hat{a}'_{ij}(\hat{u})(\hat{u}_h - \hat{u})\widehat{D}_i(\hat{u}_h - \hat{u}), \quad j = 1,\ldots,d.$$

Hence,

$$\|\hat{\varepsilon}_0\|'_{0,\infty,\widehat{\Omega}} \leqslant C\|\hat{u} - \hat{u}_h\|'^2_{0,\infty,\widehat{\Omega}}, \tag{3.24}$$

$$\|\hat{\varepsilon}_j\|'_{0,\infty,\widehat{\Omega}} \leqslant C\|\hat{u}_h\|'_{1,\infty,\widehat{\Omega}}\|\hat{u} - \hat{u}_h\|'^2_{0,\infty,\widehat{\Omega}}, \quad j = 1,\ldots,d, \tag{3.25}$$

$$\|\hat{\delta}_j\|'_{0,\infty,\widehat{\Omega}} \leqslant C\|\hat{u}-\hat{u}_h\|'_{0,\infty,\widehat{\Omega}}\|\hat{u}-\hat{u}_h\|'_{1,\infty,\widehat{\Omega}}, \quad j=1,\ldots,d. \tag{3.26}$$

Let $\hat{\rho}_h = \tilde{\hat{u}}_h - \hat{u}$, then

$$\hat{u}_h - \hat{u} = \hat{\theta}_h + \hat{\rho}_h. \tag{3.27}$$

Plugging (3.27) into (3.23) and moving all the terms about $\hat{\theta}$ to the left hand side of the equation, we have

$$\begin{aligned} &B(\hat{\theta}_h,\hat{\psi}_h) = (\hat{f}'(\hat{u})\hat{\rho}_h,\hat{\psi}_h) - \sum_{i,j=1}^{d}(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\hat{\rho}_h,\widehat{D}_j\hat{\psi}_h) + (\epsilon_0,\hat{\psi}_h) + \sum_{j=1}^{d}(\hat{\delta}_j+\epsilon_j,\widehat{D}_j\hat{\psi}_h), \quad \forall\hat{\psi}_h\in\widehat{S}_0^h,\\ &B(\hat{\theta}_h,\hat{\psi}_h) = A(\hat{\theta}_h,\hat{\psi}_h) + \sum_{i,j=1}^{d}(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\hat{\theta}_h,\widehat{D}_j\hat{\psi}_h) - (\hat{f}'(\hat{u})\hat{\theta}_h,\hat{\psi}_h). \end{aligned} \tag{3.28}$$

Let

$$\hat{\rho}_2 = \hat{u}^I - \hat{u}, \tag{3.29}$$

then

$$\hat{\rho}_h = \hat{\rho}_1 + \hat{\rho}_2. \tag{3.30}$$

With (3.11), (3.29) and Lemma 3.2, we get

$$(\hat{f}'(\hat{u})\hat{\rho}_2,\hat{\psi}_h) = -\sum_{s=1}^{m}\sum_{k=1}^{d}\hat{h}_{sk}^4\int_{\widehat{\Omega}_s}\left[\frac{1}{480}\hat{f}'(\hat{u})\hat{\psi}_h\widehat{D}_k^4\hat{u} - \frac{1}{45}\widehat{D}_k(\hat{f}'(\hat{u})\hat{\psi}_h)\widehat{D}_k^3\hat{u}\right]\mathrm{d}\hat{x} + \hat{\eta}_2(\hat{\psi}_h),$$

$$(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\hat{\rho}_2,\widehat{D}_j\hat{\psi}_h) = -\sum_{s=1}^{m}\sum_{k=1}^{d}\hat{h}_{sk}^4\int_{\widehat{\Omega}_s}\left[\frac{1}{480}\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\widehat{D}_j\hat{\psi}_h\widehat{D}_k^4\hat{u} - \frac{1}{45}\widehat{D}_k(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\widehat{D}_j\hat{\psi}_h)\widehat{D}_k^3\hat{u}\right]\mathrm{d}\hat{x} + \hat{\eta}_3(\hat{\psi}_h),$$

where

$$|\hat{\eta}_2(\hat{\psi}_h)| \leqslant C\hat{h}_0^6\|\hat{\psi}_h\|'_{2,1,\widehat{\Omega}}, \tag{3.31}$$

$$|\hat{\eta}_3(\hat{\psi}_h)| \leqslant C\hat{h}_0^6\|\widehat{D}_j\hat{\psi}_h\|'_{2,1,\widehat{\Omega}} \leqslant C\hat{h}_0^6\|\hat{\psi}_h\|'_{3,1,\widehat{\Omega}} \leqslant C\hat{h}_0^5\|\hat{\psi}_h\|'_{2,1,\widehat{\Omega}}. \tag{3.32}$$

When we construct the partition in Section 2, there are only $l$ ($l < md$) independent grid parameters $\hat{h}_1,\ldots,\hat{h}_l$ because of compatibility conditions, then

$$(\hat{f}'(\hat{u})\hat{\rho}_2,\hat{\psi}_h) = \sum_{k=1}^{l}\hat{h}_k^4\hat{M}_k(\hat{\psi}_h) + \hat{\eta}_2(\hat{\psi}_h), \tag{3.33}$$

$$(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\hat{\rho}_2,\widehat{D}_j\hat{\psi}_h) = \sum_{k=1}^{l}\hat{h}_k^4\hat{N}_k(\hat{\psi}_h) + \hat{\eta}_3(\hat{\psi}_h). \tag{3.34}$$

Here, for each $k$, $\widehat{M}_k(\hat{\psi}_h)$ is a sum of some integrations like $-\int_{\widehat{\Omega}_s}[\frac{1}{480}\hat{f}'(\hat{u})\hat{\psi}_h\widehat{D}_k^4\hat{u} - \frac{1}{45}\widehat{D}_k(\hat{f}'(\hat{u})\hat{\psi}_h)\widehat{D}_k^3\hat{u}]\mathrm{d}\hat{x}$ and $\widehat{N}_k(\hat{\psi}_h)$ is a sum of some integrations like

$$-\int_{\widehat{\Omega}_s}\left[\frac{1}{480}\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\widehat{D}_j\hat{\psi}_h\widehat{D}_k^4\hat{u} - \frac{1}{45}\widehat{D}_k(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\widehat{D}_j\hat{\psi}_h)\widehat{D}_k^3\hat{u}\right]\mathrm{d}\hat{x}.$$

By (3.20), (3.28), (3.30), (3.33), and (3.34), we can get

$$\begin{aligned} &B(\hat{\theta}_h,\hat{\psi}_h) = -\sum_{k=1}^{l}\hat{h}_k^4\left\{\sum_{i,j=1}^{d}(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\hat{\phi}_k^I,\widehat{D}_j\hat{\psi}_h) - (\hat{f}'(\hat{u})\hat{\phi}_k^I,\hat{\psi}_h) - \hat{M}_k(\hat{\psi}_h) + \hat{N}_k(\hat{\psi}_h)\right\} + \hat{\eta}(\hat{\psi}_h),\\ &\hat{\eta}(\hat{\psi}_h) = (\epsilon_0,\hat{\psi}_h) + \sum_{j=1}^{d}(\hat{\delta}_j+\epsilon_j,\widehat{D}_j\hat{\psi}_h) + (\hat{f}'(\hat{u})\hat{\eta}_1,\hat{\psi}_h) - \sum_{i,j=1}^{d}(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\hat{\eta}_1,\widehat{D}_j\hat{\psi}_h) + \hat{\eta}_2(\hat{\psi}_h) - \hat{\eta}_3(\hat{\psi}_h). \end{aligned} \tag{3.35}$$

Let

$$F_k(\hat{u},\hat{\psi}_h) = \sum_{i,j=1}^{d}(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\hat{\phi}_k^I,\widehat{D}_j\hat{\psi}_h) - (\hat{f}'(\hat{u})\hat{\phi}_k^I,\hat{\psi}_h) - \widehat{M}_k(\hat{\psi}_h) + \widehat{N}_k(\hat{\psi}_h),$$

then

$$B(\hat{\theta}_h,\hat{\psi}_h)=-\sum_{k=1}^{l}\hat{h}_k^4F_k(\hat{u},\hat{\psi}_h)+\hat{\eta}(\hat{\psi}_h). \tag{3.36}$$

Construct the following auxiliary problem: find $\hat{w}_k\in H_0^1(\widehat{\Omega})$ satisfying

$$B(\hat{w}_k,\hat{v})=F_k(\hat{u},\hat{v}),\quad \forall\hat{v}\in H_0^1(\widehat{\Omega}). \tag{3.37}$$

Because (3.12) guarantees that $B(\cdot,\cdot)$ is coercive, Lax–Milgram theorem guarantees the existence and uniqueness of $\hat{w}_k$. Let $\overline{R}_h$ denote Ritz projection with respect to $B(\cdot,\cdot)$, then (3.36) and (3.37) lead to

$$B(\hat{\theta}_h+\sum_{k=1}^{l}\hat{h}_k^4\overline{R}_h\hat{w}_k,\hat{\psi}_h)=\hat{\eta}(\hat{\psi}_h),\quad \forall\hat{\psi}_h\in\widehat{S}_0^h. \tag{3.38}$$

Using Hölder's inequality, (3.11), (3.24), (3.25), (3.26) and the finite element error estimates, we can get

$$|(\epsilon_0,\hat{\psi}_h)|\leqslant\|\hat{\varepsilon}_0\|'_{0,\infty,\widehat{\Omega}}\|\hat{\psi}_h\|'_{0,1,\widehat{\Omega}}\leqslant C\hat{h}_0^6\|\hat{\psi}_h\|'_{0,1,\widehat{\Omega}}, \tag{3.39}$$

$$|(\hat{\varepsilon}_j,\widehat{D}_j\hat{\psi}_h)|\leqslant\|\hat{\varepsilon}_j\|'_{0,\infty,\widehat{\Omega}}\|\widehat{D}_j\hat{\psi}_h\|'_{0,1,\widehat{\Omega}}\leqslant C\hat{h}_0^6\|\hat{\psi}_h\|_{1,1,\widehat{\Omega}},\quad j=1,\ldots,d, \tag{3.40}$$

$$|(\hat{\delta}_j,\widehat{D}_j\hat{\psi}_h)|\leqslant\|\hat{\delta}_j\|'_{0,\infty,\widehat{\Omega}}\|\widehat{D}_j\hat{\psi}_h\|'_{0,1,\widehat{\Omega}}\leqslant C\hat{h}_0^5\|\hat{\psi}_h\|'_{1,1,\widehat{\Omega}},\quad j=1,\ldots,d. \tag{3.41}$$

Similarly, with Hölder's inequality and (3.21), we have

$$|(\hat{f}'(\hat{u})\hat{\eta}_1,\hat{\psi}_h)|\leqslant C\hat{h}_0^{4+\alpha}|\ln\hat{h}_0|^{\frac{d-1}{d}}\|\hat{\psi}_h\|'_{0,1,\widehat{\Omega}}, \tag{3.42}$$

$$|(\hat{a}'_{ij}(\hat{u})\widehat{D}_i\hat{u}\hat{\eta}_1,\widehat{D}_j\hat{\psi}_h)|\leqslant C\hat{h}_0^{4+\alpha}|\ln\hat{h}_0|^{\frac{d-1}{d}}\|\hat{\psi}_h\|'_{1,1,\widehat{\Omega}},\quad j=1,\ldots,d. \tag{3.43}$$

By (3.31), (3.32), (3.35), (3.39), (3.40), (3.41), (3.42) and (3.43), we get

$$|\hat{\eta}(\hat{\psi}_h)|\leqslant C\hat{h}_0^{4+\beta_1}|\ln\hat{h}_0|^{\frac{d-1}{d}}\|\hat{\psi}_h\|'_{2,1,\widehat{\Omega}},\beta_1=\min(1,\alpha). \tag{3.44}$$

Let $G^z$ denote the regularized Green's function satisfying

$$B(\hat{v},G_z)=\hat{v}(Z),\quad \forall Z\in\widehat{\Omega},\hat{v}\in H_0^1(\widehat{\Omega}). \tag{3.45}$$

$G_h^z=\overline{R}_hG_z$ is the Ritz projection of $G_z$. Let $\hat{\psi}_h=G_h^z$ in (3.38). From the coercivity of $B(\cdot,\cdot)$, (3.38), (3.44) and (3.45) we can get that $\forall z\in\widehat{\Omega}$,

$$\left|\left(\hat{\theta}_h+\sum_{k=1}^{l}\hat{h}_k^4\overline{R}_h\hat{w}_k\right)(Z)\right|=\left|B\left(\hat{\theta}_h+\sum_{k=1}^{l}\hat{h}_k^4\overline{R}_h\hat{w}_k,G_z^h\right)\right|=|\hat{\eta}(G_z^h)|\leqslant C\hat{h}_0^{4+\beta_1}|\ln\hat{h}_0|^{\frac{d-1}{d}}\|G_z^h\|'_{2,1,\widehat{\Omega}}.$$

Since $Z$ is arbitrary, then Lemma 3.3 gives

$$\|\hat{\theta}_h+\sum_{k=1}^{l}\hat{h}_k^4\overline{R}_h\hat{w}_k\|'_{0,\infty,\widehat{\Omega}}\leqslant C\hat{h}_0^{4+\beta_1}|\ln\hat{h}_0|^{\frac{2d-1}{d}}.$$

Hence,

$$\hat{\theta}_h(Z)+\sum_{k=1}^{l}\hat{h}_k^4\overline{R}_h\hat{w}_k(Z)=\varepsilon,\|\varepsilon\|'_{0,\infty,\widehat{\Omega}}\leqslant C\hat{h}_0^{4+\beta_1}|\ln\hat{h}_0|^{\frac{2d-1}{d}}.$$

If $\hat{w}_k\in W_p^r(\widehat{\Omega}),r-\frac{d}{p}>0$, then

$$\hat{\theta}_h=-\sum_{k=1}^{l}\hat{h}_k^4\hat{w}_k^I+\hat{\eta}_4 \tag{3.46}$$

$$\|\hat{\eta}_4\|'_{0,\infty,\widehat{\Omega}}\leqslant C\hat{h}_0^{4+\beta}|\ln\hat{h}_0|^{\frac{2d-1}{d}}\text{ and }\beta=\min\left(\beta_1,r-\frac{d}{p}\right)=\min\left(1,\alpha,r-\frac{d}{p}\right). \tag{3.47}$$

Let

$$\hat{\psi}_k=\hat{\phi}_k-\hat{w}_k,\quad k=1,\ldots,l. \tag{3.48}$$

By (3.19), (3.20), (3.21), (3.46), (3.47), and (3.48), we finish the proof. □

**Remark 3.1.** Similar to the Remark 4 in [20], the expansion holds not only for all the grid nodes in $\widehat{\Omega}_0^h$, but also for all the edge midpoints and centers in $\widehat{\Omega}_0^h$.

## 4. Splitting extrapolation formulas at all globally fine grid points

In Section 3, we proved the multi-parameter asymptotic expansion of discrete d-quadratic iso-parametric finite element errors. Based on this expansion, in this section we will develop the splitting extrapolation formulas by applying the basic idea of splitting extrapolation. We will develop the splitting extrapolation formulas for all the nodes in the globally fine grid, not only on the coarse grid and locally fine grids. Let $\hat{u}_h^{(0)}$ and $\hat{u}_h^{(i)}$ be the approximations on $\widehat{\Omega}_0^h$ and $\widehat{\Omega}_i^h$, respectively. Let $\varepsilon = O(\hat{h}_0^{4+\beta}|\ln \hat{h}_0|^{\frac{2d-1}{d}})$.

(1) Type 0: grid points in $\widehat{\Omega}_0^h$. Suppose A is a grid point in $\widehat{\Omega}_0^h$. First, we prove the following theorem with the same idea as in [9,13,20,22].

**Theorem 4.1.**

$$\frac{16}{15}\sum_{i=1}^{l}\hat{u}_h^{(i)}(A) + \left[-\frac{16}{15}l+1\right]\hat{u}_h^{(0)}(A) - \hat{u}(A) = \varepsilon. \tag{4.49}$$

**Proof.** From (3.13), we have

$$\hat{u}_h^{(0)}(A) = \hat{u}(A) + \sum_{i=1}^{l}\hat{h}_i^4\hat{\psi}_i(A) + \varepsilon, \tag{4.50}$$

$$\hat{u}_h^{(k)}(A) = \hat{u}(A) + \sum_{\substack{i=1\\ i\neq k}}^{l}\hat{h}_i^4\hat{\psi}_i(A) + \frac{1}{16}\hat{h}_k^4\hat{\psi}_k(A) + \varepsilon, \quad k = 1,\ldots,l. \tag{4.51}$$

$\forall$ real numbers $\alpha_i, i = 1,2$, multiply (4.50) by $\alpha_1$ and (4.51) by $\alpha_2$, then add them together to get

$$\alpha_1\hat{u}_h^{(0)}(A) + \alpha_2\sum_{k=1}^{l}\hat{u}_h^{(k)}(A) = (\alpha_1+\alpha_2 l)\hat{u}(A) + \left[\alpha_1+\alpha_2(l-1)+\alpha_2\frac{1}{16}\right]\sum_{i=1}^{l}\hat{h}_i^4\hat{\psi}_i(A) + \varepsilon. \tag{4.52}$$

Let $\alpha_1+\alpha_2 l = 1, \alpha_1+\alpha_2(l-1)+\alpha_2\frac{1}{16} = 0$. Then, solving the equations for $\alpha_1$ and $\alpha_2$ and plugging them into (4.52), we finish the proof. □

From Theorem 4.1, we can get the splitting extrapolation formula for grid points in $\widehat{\Omega}_0^h$ as follows:

$$u_0(A) = \frac{16}{15}\sum_{i=1}^{l}\hat{u}_h^{(i)}(A) + \left[-\frac{16}{15}l+1\right]\hat{u}_h^{(0)}(A). \tag{4.53}$$

(2) Type 1: grid points in $\bigcup_{i=1}^{l}\widehat{\Omega}_i^h \setminus \widehat{\Omega}_0^h$. Let $A_1$ and $A_2$ be two neighboring coarse grid points. Suppose $B$ is the midpoint of $A_1A_2$ and $B \in \widehat{\Omega}_i^h \setminus \widehat{\Omega}_0^h$. First, we prove the following theorem with the same idea as in [9,13,20,22].

**Theorem 4.2.**

$$\hat{u}_h^{(i)}(B) - \frac{1}{30}\sum_{k=1}^{2}\left[\hat{u}_h^{(0)}(A_k)-\hat{u}_h^{(i)}(A_k)\right] - \frac{8}{15}\sum_{\substack{j=1\\ j\neq i}}^{l}\sum_{k=1}^{2}\left[\hat{u}_h^{(0)}(A_k)-\hat{u}_h^{(j)}(A_k)\right] = \hat{u}(B)+\varepsilon+O(\hat{h}_0^5).$$

**Proof.** Because $A_1$ and $A_2$ are coarse grid points, (4.50) and (4.51) are still true for them. Therefore, $\forall j = 1,\ldots,l, k = 1,2$,

$$\hat{u}_h^{(0)}(A_k) - \hat{u}_h^{(j)}(A_k) = \frac{15}{16}\hat{h}_j^4\hat{\psi}_j^n(A_k) + \varepsilon.$$

Then

$$\hat{h}_j^4\hat{\psi}_j^n(A_k) = \frac{16}{15}\left[\hat{u}_h^{(0)}(A_k) - \hat{u}_h^{(j)}(A_k)\right] + \varepsilon.$$

Because

$$\hat{\psi}_j^n(B) = \frac{\hat{\psi}_j^n(A_1)+\hat{\psi}_j^n(A_2)}{2} + O(\hat{h}_0),$$

we get

$$\hat{h}_j^4\hat{\psi}_j^n(B) = \frac{8}{15}\sum_{k=1}^{2}\left[\hat{u}_h^{(0)}(A_k) - \hat{u}_h^{(j)}(A_k)\right] + \varepsilon + O(\hat{h}_0^5). \tag{4.54}$$

By Remark 3.1 and (3.13), we get

$$\hat{u}_h^{(i)}(B) = \hat{u}(B) + \sum_{\substack{j=1\\ j\neq i}}^{l}\hat{h}_j^4\hat{\psi}_j(B) + \frac{1}{16}\hat{h}_i^4\hat{\psi}_i(B) + \varepsilon. \tag{4.55}$$

Plugging (4.54) into (4.55), we finish the proof. □

From Theorem 4.2, we can get the splitting extrapolation formula for grid points in $\widehat{\Omega}_i^h \setminus \widehat{\Omega}_0^h$ as follows.

$$u_1(B) = \hat{u}_h^{(i)}(B) - \frac{1}{30}\sum_{k=1}^{2}\left[\hat{u}_h^{(0)}(A_k) - \hat{u}_h^{(i)}(A_k)\right] - \frac{8}{15}\sum_{\substack{j=1\\ j\neq i}}^{l}\sum_{k=1}^{2}\left[\hat{u}_h^{(0)}(A_k) - \hat{u}_h^{(j)}(A_k)\right]. \tag{4.56}$$

(3) Type 2: Centers of rectangular elements. Suppose C is the center of a rectangular element, $A_k(k = 1,\dots,4)$ are the four vertices and $B_k(k = 1,\dots,4)$ are the midpoints of the four edges. First, $U_0(A_k)$ and $U_1(B_k)$ are computed according to (4.53) and (4.56). Then by using an incomplete bi-quadratic interpolation without term $x^2y^2$ [19,24], we get

$$u_2(C) = \frac{1}{2}\sum_{k=1}^{4}u_1(B_k) - \frac{1}{4}\sum_{k=1}^{4}u_0(A_k). \tag{4.57}$$

(4) Type 3: Centers of rectangular parallelepiped elements. Suppose $D$ is the center of a rectangular parallelepiped element, $A_k(k = 1,\dots,8)$ are the eight vertices and $B_k(k = 1,\dots,12)$ are the midpoints of the 12 edges. First, $U_0(A_k)$ and $U_1(B_k)$ are computed according to (4.53) and (4.56). Then by using an incomplete tri-quadratic interpolation without term $x^2y^2z^2, x^2y^2z, x^2yz^2, xy^2z^2, x^2y^2, x^2z^2, y^2z^2$ [19,24], we get

$$u_3(D) = \frac{1}{4}\sum_{k=1}^{12}u_1(B_k) - \frac{1}{4}\sum_{k=1}^{8}u_0(A_k). \tag{4.58}$$

## 5. Parallel/sequential algorithm

In this section, we will introduce a parallel/sequential algorithm based on the splitting extrapolation formulas in Section 5 and the idea in [9,12,20,30].

Step 1: Construct the initial domain decomposition $\overline{\Omega} = \bigcup_{i=1}^{m}\overline{\Omega}_i$ according to the dimension and interface of the problem, the shape and size of the domain, and the computers used, which satisfies the compatible conditions (1), (2) and (3) in Section 2. Obviously, by using iso-parametric or sub-parametric mapping of a high enough degree, the mapping $\Psi_i$ may be constructed, see [24] and reference therein.
Step 2: Construct the uniform cuboid partition $\mathfrak{I}_i^h(i = 1,\dots,m)$ for each $\widehat{\Omega}_i$ with independent grid parameters $\hat{h}_1,\dots,\hat{h}_l$, which satisfies the compatible conditions (4), (5), (6) and (7) in Section 2.
Step 3: Compute $\hat{u}_h^{(i)}, i = 0,\dots,l$ by using the standard finite element method in parallel/sequentially. All processors call the same subroutines with different input parameters if we use parallel computation.
Step 4: Implement (4.53), (4.56), (4.57) and (4.58) to all the grid nodes, the edge middle points, the centers of the rectangular elements and centers of rectangular parallelepiped elements of the coarse grid $\widehat{\Omega}_0^h$ by using the results from all processors.

**Remark 5.1.** The following is a pseudo code by using MPI for steps 3 and 4 in parallel computation. Suppose the independent parameters for the coarse grid form a vector $h = (h_1,\dots,h_l)$ and the name of the subroutine to compute $\hat{u}_h^{(i)}, i = 0,\dots,l$ is solve_uh.

```
INCLUDE 'mpif.h'
CALL MPI_INIT(error_inf)
CALL MPI_COMM_SIZE(MPI_COMM_WORLD, total_processors, error_inf)
CALL MPI_COMM_RANK(MPI_COMM_WORLD, i, error_inf)
l = total_processors − 1
If (i.eq. 0) THEN
  CALL solve_uh(h, u_0)
ELSE
  h_fined(:)=h(:)
  h_fined(i)=h(i)/2
```

```
CALL solve_uh(h_fined, u_i)
ENDIF
If (i.eq. 0) THEN
  DO i=1, l
    CALL MPI_RECV(receive u_i from processor i)
  ENDDO
ELSE
  CALL MPI_SEND(send u_i to processor 0)
ENDIF
IF (i.eq. 0) THEN
  CALL Extrapolation(u_final,u_0,...,u_l)/* Implement (4.53), (4.56), (4.57) and (4.58) */
ENDIF
CALL MPI_FINALIZE(error_inf)
END
```

**Remark 5.2.** In step 3, if we construct the domain decomposition proportionally and design independent variables properly, the processors computing $\hat{u}_h^{(i)}(i = 1, \dots, l)$ can have almost the same load. For example, for domain $\Omega = [0,2] \times [0,2]$, we construct domain decomposition as $\Omega = \bigcup_{i=1}^4 \Omega_i, \Omega_1 = [0,1] \times [0,1], \Omega_2 = [0,1] \times [1,2], \Omega_3 = [1,2] \times [0,1], \Omega_4 = [1,2] \times [1,2]$ and design independent mesh parameters as follows. $h_1$ is the horizontal mesh parameter of $\Omega_1$ and $\Omega_2$. $h_2$ is the horizontal mesh parameter of $\Omega_3$ and $\Omega_4$. $h_3$ is the vertical mesh parameter of $\Omega_1$ and $\Omega_3$. $h_4$ is the vertical mesh parameter of $\Omega_2$ and $\Omega_4$. If we take $h_i = \frac{1}{n}, i = 1, \dots, 4$, then all the four processors computing $u_h^{(i)}(i = 1, \dots, 4)$ work on a mesh with $(2n+1)(3n+1)$ mesh nodes. Therefore, their loads are balanced.

**Remark 5.3.** For step 3, if we compute $\hat{u}_h^{(i)}, i = 0, \dots, l$ sequentially, then we can save a lot of memory. After we compute each $\hat{u}_h^{(i)}$ by using the standard finite element method, we only need to save the final results and can deallocate most of the memory. Since all the $\hat{u}_h^{(i)}, i = 0, \dots, l$ are on the coarse grid or locally fined grids, then the required memory is much less than that of the globally fine grid. However, we finally get an approximation with high accuracy on the globally fine grid by using the splitting extrapolation formulas.

## 6. A posteriori error estimates

In this section, we present some a posteriori error estimates. Suppose $A$ is a grid point in $\widehat{\Omega}_0^h$. First, we have an a posteriori error estimate for the approximation $\hat{u}_h^{(k)}(A)$ $(k = 0, \dots, l)$ as follows:

**Theorem 6.1.**

$$|\hat{u}_h^{(0)}(A) - \hat{u}(A)| \leqslant \frac{16}{15} \sum_{j=1}^{l} |\hat{u}_h^{(0)}(A) - \hat{u}_h^{(j)}(A)| + \varepsilon, \tag{6.59}$$

$$|\hat{u}_h^{(k)}(A) - \hat{u}(A)| \leqslant \frac{16}{15} \sum_{\substack{j=1 \\ j \neq k}}^{l} |\hat{u}_h^{(0)}(A) - \hat{u}_h^{(j)}(A)| + \frac{1}{15} |\hat{u}_h^{(0)}(A) - \hat{u}_h^{(k)}(A)| + \varepsilon, \quad k = 1, \dots, l. \tag{6.60}$$

**Proof.** Using (4.50) and (4.51), $\forall j = 1, \dots, l$, we get

$$\hat{u}_h^{(0)}(A) - \hat{u}_h^{(j)}(A) = \frac{15}{16} \hat{h}_j^4 \hat{\psi}_j(A) + \varepsilon. \tag{6.61}$$

Thus,

$$\hat{h}_j^4 \hat{\psi}_j(A) = \frac{16}{15} \left[ \hat{u}_h^{(0)}(A) - \hat{u}_h^{(j)}(A) \right] + \varepsilon. \tag{6.62}$$

Plugging (6.59) and (6.63) back into (4.50) and (4.51) and using the triangular inequality, we finish the proof. □

Second, we have an a posteriori error estimate for the average of $\hat{u}_h^{(k)}(A)(j = 1, \dots, l)$ as follows.

**Theorem 6.2.**

$$\left| \frac{1}{l} \sum_{k=1}^{l} \hat{u}_h^{(k)}(A) - \hat{u}(A) \right| \leqslant \left( \frac{16}{15} l - 1 \right) \left| \frac{1}{l} \sum_{k=1}^{l} \hat{u}_h^k(A) - \hat{u}_h^0(A) \right| + \varepsilon. \tag{6.63}$$

**Proof.** Using the triangular inequality and (4.49), we finish the proof. □

## 7. Numerical experiments

In this section, we will present two numerical examples to illustrate the features of splitting extrapolation in this article. We will see that our method is efficient for solving discontinuous problems if we regard the interfaces of the problems as the interfaces of the initial domain decomposition.

**Example 1.** Consider a semi-linear elliptic interface equation

$$\begin{cases} -\bigtriangledown(a(x,y)\bigtriangledown u) = f(x,y,u) & \text{on } \Omega \\ u(x,y) = g(x,y) & \text{on } \partial\Omega \end{cases}$$

together with the jump conditions on the interface $\Gamma = \{(x,y) : x = 1, 0 \leqslant y \leqslant 1\}$:

$$[u]|_\Gamma = 0, \left[a(x,y)\frac{\partial u}{\partial n}\right]|_\Gamma = 0, a(x,y) = \begin{cases} r, & \text{on } \Omega\bigcap\{x < 1\}, \\ 1, & \text{on } \Omega\bigcap\{x \geqslant 1\}. \end{cases}$$

An example of this problem is a simple electrostatic field problem in which two objects of different materials stick to each other. Let $\Omega$ be a curved quadrangle. Its bottom boundary is a line through $P_1 = (0,0)$ and $P_2 = (2,0)$ while its top boundary is a line through $P_4 = (0,1)$ and $P_3 = (2,1)$. The left side boundary is a parabola through points $P_1$, $P_8 = (-0.25, 0.5)$ and $P_4$. The right side boundary is a parabola through points $P_2$, $P_6 = (2.25, 0.5)$ and $P_3$. Let $P_5 = (1,0), P_7 = (1,1), P_9 = (1, \frac{1}{2})$ and

$$f(x,y,u) = \begin{cases} u + 15r(r+1)(3x-2)y(y-1) - 30rx + 15r(r+1)x(x-1)^2 - 15xy(y-1) \\ +7.5(r+1)(x-1)^2xy(y-1), \quad \text{on } \Omega\bigcap\{x<1\}, \\ u + 15(r+1)(3x-2)y(y-1) - 30(rx+1-r) + 15(r+1)x(x-1)^2 - 15rxy(y-1) \\ -15(1-r)y(y-1) + 7.5(r+1)(x-1)^2xy(y-1), \quad \text{on } \Omega\bigcap\{x \geqslant 1\}. \end{cases}$$

Here, $r = 0.5$. First, we construct an initial domain decomposition $\overline{\Omega} = \bigcup_{s=1}^2 \Omega_s$ where $\Omega_1 = \Omega\bigcap\{x < 1\}$ and $\Omega_2 = \Omega\bigcap\{x \geqslant 1\}$. With the d-quadratic iso-parametric mapping, $\Omega$, $\Omega_1$, and $\Omega_2$ are mapped to $\widehat{\Omega} = [0,2]\times[0,1]$, $\widehat{\Omega}_1 = [0,1)\times[0,1]$, and $\widehat{\Omega}_2 = [1,2]\times[0,1]$ separately. Then we design three independent step sizes as follows: $h_i (i = 1,2)$ are the step sizes of $\widehat{\Omega}_i (i = 1,2)$ in the $x$-direction; $h_3$ is the step size in the $y$-direction. We use Newton iteration for the nonlinear system. Let $h_i = \frac{1}{4}\,(i = 1,2,3)$. Some results are shown in Table 1. In order to get the splitting extrapolation solution on the globally fine grid, we only need to apply the standard finite element method on the coarse grid and the locally fine grids. Therefore, we do not compute the standard finite element solution at the globally fine grid points which are not the grid points of the coarse grid or locally fine grids. Let ** denote these unknown results in the following tables. In our examples, the coarse grid is the grid with the step size $h_i = \frac{1}{4}(i = 1,2,3)$ and the globally fine grid is the grid with the step size $h_i = \frac{1}{8}(i = 1,2,3)$. Error of FE is the error of the standard finite element approximation. Error of SEM is the error of the splitting extrapolation solution. Max error is the maximum error of all grid points. In Table 2, we show the maximum values of a posteriori error estimates at all coarse grid points, which are discussed in Section 7. Let APE1 be the maximum value of a posteriori error estimate in (6.59), APE2, APE3 and APE4 be that of a posteriori error estimate in (6.60) for $k = 1,2,3$ separately and APE5 be that of a posteriori error estimate in (6.63) (See Table 3).

**Example 2.** Consider a quasi-linear elliptic equation for

$$\begin{cases} -\bigtriangledown((1+u^2)\bigtriangledown u) = f(x,y,u) & \text{on } \Omega, \\ u(x,y) = g(x,y) & \text{on } \partial\Omega \end{cases}$$

**Table 1**
Some numerical results for Example 1

| Grid points | Point type | Error of FE | Error of SEM |
|---|---|---|---|
| (0.0293, 0.8750) | Type 0 | $+1.7041\times10^{-6}$ | $+2.6020\times10^{-8}$ |
| (1.0000, 0.1250) | Type 0 | $+5.3739\times10^{-5}$ | $-7.1147\times10^{-8}$ |
| (−0.0029, 0.6250) | Type 1 | $+1.1420\times10^{-5}$ | $-1.0005\times10^{-8}$ |
| (1.0000, 0.5625) | Type 1 | $+9.5820\times10^{-5}$ | $+8.0716\times10^{-7}$ |
| (0.6399, 0.8125) | Type 2 | ** | $+1.0381\times10^{-5}$ |
| (−0.0803, 0.1875) | Type 2 | ** | $-3.1115\times10^{-4}$ |
| Max error on coarse grid | | $+9.8016\times10^{-5}$ | $-1.8928\times10^{-6}$ |
| Max error on fine grid | | ** | $+6.5470\times10^{-4}$ |

**Table 2**
Some numerical results for a posteriori error estimates of Example 1

| APE1 | APE2 | APE3 | APE4 | APE5 |
|---|---|---|---|---|
| $2.5077\times10^{-4}$ | $1.7968\times10^{-4}$ | $8.7778\times10^{-5}$ | $2.4975\times10^{-4}$ | $6.8141\times10^{-5}$ |

**Table 3**
Some numerical results for Example 2

| Grid points | Point type | Error of FE | Error of SEM |
|---|---|---|---|
| (0.5371, 0.3750) | Type 0 | $+6.6936 \times 10^{-4}$ | $-6.1788 \times 10^{-7}$ |
| (1.0000, 0.7500) | Type 0 | $+7.5411 \times 10^{-4}$ | $-2.7759 \times 10^{-6}$ |
| (0.3320, 0.2500) | Type 1 | $+6.1854 \times 10^{-4}$ | $+5.5820 \times 10^{-7}$ |
| (1.0000, 0.6875) | Type 1 | $+9.7828 \times 10^{-5}$ | $-1.5739 \times 10^{-6}$ |
| (1.9924, 0.0625) | Type 2 | ** | $-2.6682 \times 10^{-6}$ |
| (0.9241, 0.6875) | Type 2 | ** | $-7.3853 \times 10^{-5}$ |
| Max error on coarse grid | | $+7.8225 \times 10^{-4}$ | $-1.5310 \times 10^{-5}$ |
| Max error on fine grid | | ** | $-1.0632 \times 10^{-4}$ |

**Table 4**
Some numerical results for a posteriori error estimates of Example 2

| APE1 | APE2 | APE3 | APE4 | APE5 |
|---|---|---|---|---|
| $7.9946 \times 10^{-4}$ | $7.5841 \times 10^{-4}$ | $7.5841 \times 10^{-4}$ | $2.2837 \times 10^{-4}$ | $5.1345 \times 10^{-4}$ |

An example of this problem is a heat conduction problem whose thermal conductivity depends on the temperature. Let $f(x,y,u) = \frac{5}{4}\pi^2(u+u^3) - \frac{\pi^2}{2}\sin^3(\pi y)\sin(\frac{\pi x}{2})\cos^2(\frac{\pi x}{2}) - 2\pi^2\sin^3(\frac{\pi x}{2})\sin(\pi y)\cos^2(\pi y)$. Let $\Omega$ be the same curved quadrangle as in Example 1. The initial domain decomposition and the design of independent step sizes are the same as in Example 1.We use Newton iteration for the nonlinear system. Let $h_i = \frac{1}{4}(i = 1,2,3)$. Some results are shown in Table 2. In Table 4, we also show the maximum values of a posteriori error estimates at all coarse grid points for Example 2.

## 8. Conclusions

The splitting extrapolation formulas are just some linear combinations and can be easily implemented. They generate an approximation with higher accuracy on a globally fine grid while only requiring some approximations from a set of smaller discrete subproblems on different coarser grids. Because these subproblems are independent of each other and have similar scales, the method is naturally parallel and also possesses a high degree of parallelism. Additionally, the multi-parameter expansion only requires the local smoothness of the solutions, i.e., the smoothness of the solutions in each sub-domain. Therefore, splitting extrapolation is efficient for solving discontinuous problems if we regard the interfaces of the problems as the interfaces of the initial domain decomposition. The numerical examples above verify our theoretical analysis.

## References


[1] F.G. Basombrio, L. Guarracino, M.J. Venere, A non-iterative algorithm based on richardson's extrapolation. application to groundwater flow modelling, Int. J. Numer. Meth. Eng. 65 (7) (2006) 1088–1112.
[2] H. Blum, Q. Lin, R. Rannacher, Asymptotic error expansion and richardson extrapolation for linear finite elements, Numer. Math. 49 (1) (1986) 11–37.
[3] S.C. Brenner, L.R. Scott, The mathematical theory of finite element methods, Texts in Applied Mathematics, second ed., vol. 15, Springer-Verlag, New York, 2002.
[4] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[5] M.D. Dramicanin, V. Djokovic, S. Galovic, Theory of photothermal effects in thermally inhomogeneous solids with constant effusivity, J. Phys. D: Appl. Phys. 33 (14) (2000) 1736–1738.
[6] G. Fairweather, Q. Lin, Y. Lin, J. Wang, S. Zhang, Asymptotic expansions and richardson extrapolation of approximate solutions for second order elliptic problems on rectangular domains by mixed finite element methods, SIAM J. Numer. Anal. 44 (3) (2006) 1122–1149.
[7] V.F. Formalev, Heat and mass transfer in anisotropic bodies, High Temp. 39 (5) (2001) 753–774.
[8] J. Frehse, R. Rannacher, Eine $l$1-fehlerabschStzung fnr diskrete grundlsungen in der methode der finiten elemente(german), Bonn. Math. Schrift 89 (1976) 92–114.
[9] X.-M. He, T. Lü, Splitting extrapolation method for solving second order parabolic equations with curved boundaries by using domain decomposition and d-quadratic isoparametric finite elements, Int. J. Comput. Math. 84 (6) (2007) 767–781.
[10] J. Huang, T. Lü, Splitting extrapolations for solving boundary integral equations of linear elasticity dirichlet problems on polygons by mechanical quadrature methods, J. Comput. Math. 24 (1) (2006) 9–18.
[11] S. Jia, H. Xie, X. Yin, S. Gao, Approximation and eigenvalue extrapolation of biharmonic eigenvalue problem by nonconforming finite element methods, Numer. Meth. Part. Differen. Equat. 24 (2) (2008) 435–448.
[12] X. Liao, A. Zhou, A multi-parameter splitting extrapolation and a parallel algorithm for elliptic eigenvalue problem, J. Comput. Math. 16 (3) (1998) 213–220.
[13] C.B. Liem, Tao Lü, T.M. Shin, The splitting extrapolation method, a new technique in numerical solution of multidimensional problems, with a preface by Zhong-ci Shi, Series on Applied Mathematics, World Scientific, Singapore, 1995.
[14] Q. Lin, J.F. Lin, Extrapolation of the bilinear element approximation for the poisson equation on anisotropic meshes, Numer. Meth. Part. Differen. Equat. 23 (5) (2007) 960–967.
[15] Q. Lin, T. Lü, The splitting extrapolation method for multidimensional problem, J. Comput. Math. 1 (1983) 45–51.
[16] Q. Lin, Q.D. Zhu, Undirectional extrapolations of finite difference and finite elements, J. Eng. Math. 1 (1984) 1–12.
[17] T. Lin, Y. Lin, M. Rao, S. Zhang, Petrov-galerkin methods for linear volterra integro-differential equations, SIAM J. Numer. Anal. 38 (3) (2000) 937–963.
[18] T. Lü, Correction and splitting extrapolation methods for the collocation solutions of two point boundary value problems, Adv. Math. 16 (1987) 39–391 (in Chinese).

[19] T. Lü, Y. Feng, Splitting extrapolation based on domain decomposition for finite element approximations, Sci. China Ser. E 40 (2) (1997) 144–155.
[20] T. Lü, J. Lu, Splitting extrapolation for solving second order elliptic systems with curved boundary in rd by using d-quadratic isoparametric finite element, Appl. Numer. Math. 40 (4) (2002) 467–481.
[21] T. Lü, T.M. Shih, C.B. Liem, An analysis of the splitting extrapolation for multidimensional problem, Syst. Sci. Math. Sci. 3 (3) (1990) 261–272.
[22] T. Lü, T.M. Shin, C.B. Liem, Splitting Extrapolation and Combination techniques, Scientific Press, Beijing, 1998 (in Chinese).
[23] G. Marchuk, V. Shaidurov, Difference methods and their extrapolations, translated from the Russian, Applications of Mathematics, Springer-Verlag, New York, 1983.
[24] P. Neittaanmaki, Q. Lin, Acceleration of the convergence in finite difference methods by predictor corrector and splitting extrapolation methods, J. Comput. Math. 5 (1987) 181–190.
[25] K. Rahul, S.N. Bhattacharyya, One-sided finite-difference approximations suitable for use with richardson extrapolation, J. Comput. Phys. 219 (1) (2006) 13–20.
[26] R. Rannacher, R. Scott, Some optimal error estimates for piecewise linear finite element approximations, Math. Comp. 38 (158) (1982) 437–445.
[27] U. Rüde, Book review: the splitting extrapolation method (C.B. Liem, T. Lü and T.M. Shih), SIAM Rev. 39 (1997) 161–162.
[28] U. Rüde, A. Zhou, Multi-parameter extrapolation methods for boundary integral equations. numerical treatment of boundary integral equations, Adv. Comput. Math. 9 (1-2) (1998) 173–190.
[29] A. Sidi, Practical extrapolation methods, theory and applications, Cambridge Monographs on Applied and Computational Mathematics, vol. 10, Cambridge University Press, Cambridge, 2003.
[30] A. Zhou, C.B. Liem, T.M. Shih, T. Lü, A multi-parameter splitting extrapolation and a parallel algorithm, Syst. Sci. Math. Sci. 10 (3) (1997) 253–260.
[31] C. Zhu, Q. Lin, The hyperconvergence theory of finite elements, Hunan Science and Technology Publishing House, Changsha, 1989 (in Chinese).